Scattering for wave maps exterior to a ball

A. L., W. Schlag.,
http://www.math.uchicago.edu/~alawrie

UIUC, February 28, 2012

A brief introduction to wave maps:

- Definition: Formally, wave maps are critical points of the Lagrangian

$$\mathcal{L}(u, \partial u) = \int_{\mathbb{R}^{1+d}} \eta^{\alpha\beta} \langle \partial_\alpha u, \partial_\beta u \rangle_g \, dt \, dx$$

  where $u : (\mathbb{R}^{1+d}, \eta) \to (M, g)$. Here, η is the Minkowski metric on $\mathbb{R}^{1+d}$ and (M, g) is a Riemannian manifold.

- Intrinsic Formulation: Critical points of $\mathcal{L}$ satisfy the Euler-Lagrange equation

$$\eta^{\alpha\beta} D_\alpha \partial_\beta u = 0$$

- Extrinsic Formulation: If $M \hookrightarrow \mathbb{R}^N$ is embedded, critical points are characterized by

$$\Box u \perp T_u M$$

The Cauchy problem

Cauchy problem:

- Intrinsic Formulation: In local coordinates on (M, g), the Cauchy problem for wave maps is

$$\Box u^k = -\eta^{\alpha\beta}\Gamma^k_{ij}(u)\partial_\alpha u^i \partial_\beta u^j$$
$$(u, \partial_t u)|_{t=0} = (u_0, u_1)$$

  where Γ^k_{ij} are the Christoffel symbols on TM.

- Extrinsic Formulation: In the embedded case, the Cauchy problem becomes

$$\Box u = \eta^{\alpha\beta} S(u)(\partial_\alpha u, \partial_\beta u)$$
$$(u, \partial_t u)|_{t=0} = (u_0, u_1)$$

  where S is the second fundamental form of the embedding.

Energy conservation and scaling

- Conservation of energy: Wave maps exhibit a conserved energy

$$E(u, \partial_t u)(t) = \int_{\mathbb{R}^d} (|\partial_t u|_g^2 + |\nabla u|_g^2)\, dx = \text{const.}$$

- Scaling invariance: Wave maps are invariant under the scaling $u(t,x) \mapsto u(\lambda t, \lambda x)$.
- Criticality: The scaling invariance implies that the Cauchy problem is $\dot{H}^s \times \dot{H}^{s-1}$ critical for $s = \frac{d}{2}$, energy critical when $d = 2$ and energy supercritical for $d > 2$.

Equivariant Wave Maps

Equivariant wave maps: In the presence of symmetries, e.g., $M = S^d$, one can require

$$u \circ \rho = \rho^\ell \circ u$$

where $\rho \in SO(d)$ acts on $\mathbb{R}^d$ (resp. S^d) by rotation. The action on S^d is rotation is about a fixed axis.

Foundational works:

- Shatah (1988): finite time blow-up (self-similar) for wave maps $u : \mathbb{R}^{1+d} \to S^d$ for $d \geq 3$.
- Christodoulou, Tahvildar-Zadeh (1993): Global theory for targets satisfying a convexity condition.
- Shatah, Tahvildar-Zadeh (1994): Local theory, generalization of Shatah blow-up to rotationally symmetric, non-convex targets.

Exterior Wave Maps

Issue at hand: Global well-posedness and scattering for 3d equivariant wave maps exterior to a ball.

Exterior model: We consider

$$u : \mathbb{R}_t \times (\mathbb{R}^3 \setminus B) \to S^3$$

with the Dirichlet boundary condition $u(\partial B) =$ north pole, and $B = B(0,1)$. Fixing equivariance class $\ell = 1$ we can write

$$u : (t, r, \omega) \mapsto (\psi(t, r), \omega) \mapsto (\sin(\psi(t, r)) \cdot \omega\, , \cos(\psi(t, r)))$$

where (r, ω) are polar coordinates on $\mathbb{R}^3$ and ψ measures the azimuth angle from the north pole on S^3.

1-equivariant exterior Cauchy problem

Cauchy problem in the exterior setting:

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \\ &\psi(t,1) = 0 \quad \forall t \geq 0 \\ &\vec{\psi}(0) := (\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1) \end{aligned} \tag{1}$$

Conserved energy:

$$\mathcal{E}(\vec{\psi}) = \int_1^\infty \left[\frac{1}{2}(\psi_t^2 + \psi_r^2) + \frac{\sin^2\psi}{r^2}\right] r^2\, dr$$

1-equivariant exterior Cauchy problem

Cauchy problem in the exterior setting:

$$
\begin{aligned}
&\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \qquad (1)\\
&\psi(t,1) = 0 \quad \forall t \geq 0\\
&\vec{\psi}(0) := (\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1)
\end{aligned}
$$

Conserved energy:

$$
\mathcal{E}(\vec{\psi}) = \int_1^\infty \left[\frac{1}{2}(\psi_t^2 + \psi_r^2) + \frac{\sin^2\psi}{r^2}\right] r^2\, dr
$$

- Finite energy + continuous dependence on a time interval I $\implies \psi(t,\infty) = n\pi$ for some $n \in \mathbb{N}$, for every $t \in I$. $\implies$ every wave map has a fixed topological degree.
- The natural space for the solution in the energy class defined by $n = 0$ is $\mathcal{H} := \dot{H}_0^1 \times L^2(1,\infty)$ with the norm

$$
\|\vec{\psi}\|_{\mathcal{H}}^2 = \int_1^\infty (\psi_t^2 + \psi_r^2)\, r^2\, dr
$$

Why is the exterior 3d problem interesting? Removing a ball gives rise to a family of nontrivial harmonic maps Q_n indexed by the topological degree n.

Harmonic maps: A "degree n" harmonic map in this context is a solution to the following problem:

$$Q_{rr} + \frac{2}{r} Q_r = \frac{\sin(2Q)}{r^2} \tag{2}$$
$$Q(1) = 0, \quad Q(\infty) = n\pi$$

- $n = 0$: In the zero topological class we have $Q \equiv 0$.
- $n \geq 1$: After the change of variables $t = \log(r)$, $x(t) := Q(r)$, set $y = \dot{x}$ and (2) becomes the autonomous system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + \sin(2x) \end{pmatrix} \tag{3}$$
$$x(0) = 0, \quad x(\infty) = n\pi$$

3d Harmonic Map phase portrait

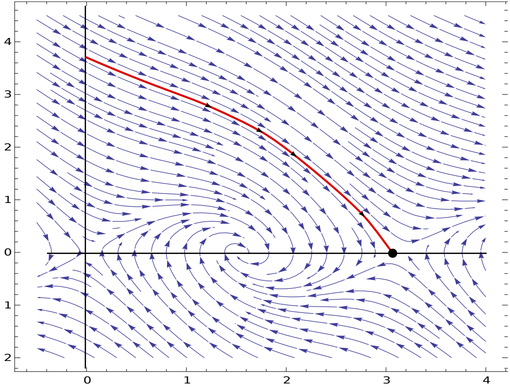

Figure: The red flow line is a depiction of the harmonic map Q_1 which connects the north pole to the south pole, i.e., $Q(1) = 0$ and $Q(\infty) = \pi$

- This is the equation of a damped pendulum.
- $3d$ non-exterior problem there are no harmonic maps...

2d Harmonic Maps?

2d Harmonic maps equation: In 2d, the exterior harmonic map equation reduces to the equation of a simple pendulum

$$\ddot{x} = \frac{1}{2}\sin(2x), \quad x(0) = 0, \quad x(\infty) = n\pi$$

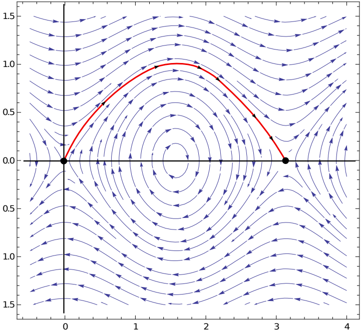

Figure: The red flow line is a depiction of the harmonic map Q for the non-exterior problem which connects the north pole to the south pole.

Harmonic Maps Summary

	2d	3d
full problem	Unique harmonic map Q with $Q(0) = 0$ (up to scaling)	No nontrivial harmonic maps
exterior problem	No nontrivial harmonic maps	Family Q_n of harmonic maps indexed by topological degree

Back to the 3d exterior model:

Soliton Resolution Conjecture: Informally, this conjecture asserts that given "generic" initial data for a dispersive equation with global solutions, the long term behavior of the global evolution should eventually resolve into a superposition of solitons and a radiation component that decays.

- The $3d$ exterior wave map problem was proposed by Bizon, Chmaj, Maliborksi (2011), as a simple model to study relaxation to the ground states (given by the harmonic maps).
- Removing a ball breaks the scaling symmetry!
- B-C-M make the simple observation that removing the origin effectively renders the $3d$ Cauchy problem subcritical. ($3d$ equiv. wave maps to the sphere are supercritical in non-exterior case). Global existence becomes a triviality.
- Numerical simulations suggest that in each energy class defined by the topological class, $\psi(\infty) = n\pi$, every solution scatters to the unique harmonic map Q_n in that class.

Main Results

Theorem 1 (L, Schlag, 2011)

$n = 0$, $Q_0 = 0$. For any smooth energy data $(\psi_0, \psi_1) \in \mathcal{H}$, there exists a global smooth evolution $\vec{\psi}$ to (1). Furthermore, $\vec{\psi}$ scatters to 0 in the sense that the energy of $\vec{\psi}$ on any arbitrary, but compact region vanishes as $t \to \infty$.

Theorem 2 (L, Schlag, 2011)

$n \geq 1$, $Q_n = Q$. There exists $\varepsilon > 0$ such that for all smooth data $(\psi_0, \psi_1) \in \mathcal{H}_n$ such that

$$\|(\psi_0, \psi_1) - (Q, 0)\|_{\mathcal{H}} < \varepsilon$$

the unique solution ψ to (1) with data (ψ_0, ψ_1) exists globally in time and scatters to Q as $t \to \infty$.

Scattering

Scattering: Here scattering can be phrased as follows: There exists (φ, φ_t) such that

$$(\psi, \psi_t) = (Q_n, 0) + (\varphi, \varphi_t) + o_{\mathcal{H}}(1) \quad \text{as} \quad t \to \infty$$

where $\vec{\varphi}$ solves the linearized equation

$$\begin{aligned} &\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{2}{r^2}\varphi = 0 \\ &\varphi(t, 1) = 0 \end{aligned}$$

- Cote, Kenig, Merle (2008) prove scattering for $2d$ wave maps (non-exterior) for data with energy slightly above $\mathcal{E}(Q, 0)$ via the celebrated Kenig-Merle concentration-compactness/rigidity method, Kenig, Merle (2006 Invent.), (2008 Acta.). We also employ the Kenig-Merle method here.

Kenig-Merle method: We outline the proof of Theorem 1. Let

$$\mathcal{S}_+ = \{(\psi_0, \psi_1) \in \mathcal{H} \,|\, \vec{\psi}(t) \text{ exists globally and scatters as } t \to +\infty\}$$

We claim that $\mathcal{S}_+ = \mathcal{H}$. This is proved via the following outline:

- (Small data result): Small data global existence and scattering, proving $\mathcal{S}_+$ is not empty.
- (Concentration Compactness): If Theorem 1 fails, i.e., if $\mathcal{S}_+ \neq \mathcal{H}$, then there exists a nonzero energy solution $\vec{\psi}$ to (1) (called the critical element) such that the trajectory

  $$K_+ = \{\vec{\psi}(t) \,|\, t \geq 0\}$$

  is precompact in $\mathcal{H}$.
- (Rigidity Argument): If a global evolution $\vec{\psi}$ has the property that the trajectory, K_+, is pre-compact in $\mathcal{H}$, then $\psi \equiv 0$.

Small Data Scattering

Small data scattering: The small data global existence and scattering result follows from the Smith-Sogge Strichartz estimates for 5d linear exterior wave equations (2000 CPDE) after the following reduction: Set $u := \frac{\psi}{r}$. Then u satisfies the following equation:

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{\sin(2ru) - 2ru}{r^3} = 0 \tag{4}$$
$$u(1, t) = 0$$

- By Hardy's inequality the map $\psi \mapsto \frac{\psi}{r}$ defines an isomorphism between $\mathcal{H}$ and $\dot{H}^1_0 \times L^2(\mathbb{R}^5 \setminus B)$, hence a small data global existence and scattering result for (4) implies the same result for (1) in $\mathcal{H}$.
- As usual, one can show that a solution u scatters to a free wave $\iff \|u\|_S < \infty$ where S is a suitably chosen Strichartz norm. In this case, $S = L^3_t L^6_x$.

Concentration Compactness

Concentration Compactness:

- small data theory $\Longrightarrow \mathcal{S}_+$ contains a small ball around zero. Hence, if Theorem 1 fails, there is a bounded sequence of data $\vec{u}_n := (u_n^0, u_n^1) \in \mathcal{H}$ such that

$$\|\vec{u}_n\|_{\mathcal{H}} \to E_* > 0, \quad \text{and} \quad \|u_n\|_S \to \infty$$

  One assumes that E_* is minimal with this property.
- Naively, we would like to "pass to the limit" in the u_n and obtain an element u_* with $\|\vec{u}_*\|_{\mathcal{H}} = E_*$ and $\|u_*\|_S = \infty$.
- However, the symmetries of the equation present an obstacle to compactness. Namely,
  1. the u_n can be arbitrarily translated in time.
  2. the u_n might split into individual waves which become arbitrarily separated in space-time as $n \to \infty$.

Concentration Compactness (continued)

Bahouri-Gerard Decomposition

$\{u_n\}$ a seq. of free radial waves bounded in $\mathcal{H} = \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$. Passing to a subsequence, $\exists$ a seq. of free solutions v^j bounded in $\mathcal{H}$, and seq.'s of times $t^j_n \in \mathbb{R}$ such that for γ^k_n defined by

$$u_n(t) = \sum_{1 \le j < k} v^j(t + t^j_n) + \gamma^k_n(t) \tag{5}$$

we have for any $j < k$, $\vec{\gamma}^k_n(-t^j_n) \rightharpoonup 0$ weakly in $\mathcal{H}$ as $n \to \infty$, $\lim_{n\to\infty} |t^j_n - t^k_n| = \infty$ and the errors γ^k_n vanish asymptotically

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma^k_n\|_{(L^\infty_t L^p_x \cap L^3_t L^6_x)(\mathbb{R}\times\mathbb{R}^5_*)} = 0 \quad \forall\, \frac{10}{3} < p < \infty \tag{6}$$

Moreover, we have orthogonality of the free energy

$$\|\vec{u}_n\|^2_{\mathcal{H}} = \sum_{1 \le j < k} \|\vec{v}^j\|^2_{\mathcal{H}} + \|\vec{\gamma}^k_n\|^2_{\mathcal{H}} + o(1) \quad as\, n \to \infty \tag{7}$$

Concentration Compactness (continued)

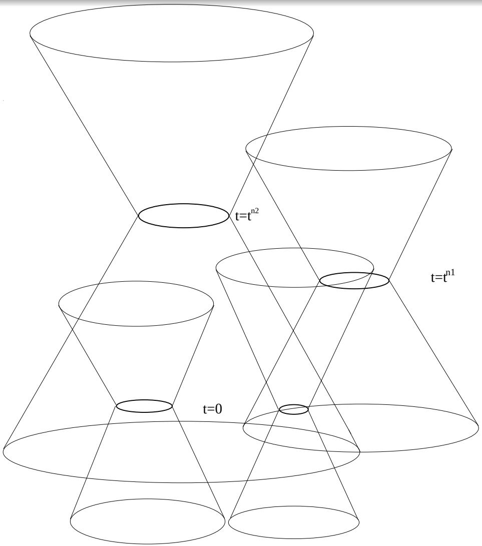

Figure: a schematic description of the concentration-compactness decomposition

Concentration Compactness (continued)

- The minimality of E_* allows one to conclude that for our sequence $\{\vec{u}_n\}$ there can be only one non-vanishing profile v^j, say, v^1.
- Indeed, the general idea is that if there were two nonzero profiles v^1 and v^2, one can conclude via the orthogonality of the energies that the corresponding non-linear profiles U^1 and U^2 each have energy less than E_* which means that U^1 and U^2 both scatter as $t \to \infty$ with uniformly controlled S norms.
- A perturbation lemma now allows one to conclude the same for the u_n which is a contradiction.
- This allows us to obtain the limiting "critical element", u_*, with $\|\vec{u}_*\|_{\mathcal{H}} = E_*$ and $\|u_*\|_S = \infty$.
- The pre-compactness of the trajectory $K_+ = \{u_*(t) \mid t \geq 0\}$ is then obtained via another application of Bahouri-Gerard.

Rigidity argument

Rigidity: The concentration compactness procedure produces a nonzero solution ψ with a compact trajectory, $K_+ = \{\vec{\psi}(t) \mid t \geq 0\}$, in the event that Theorem 1 fails. The goal now is to show that any such solution is identically zero, which is a contradiction.

- One should note that in contrast to the $2d$ scattering result of Cote, Kenig, Merle we do not need an upper bound on the energy to carry out a rigidity argument.
- Indeed, we show that the nonlinear functional $\mathcal{L}$ associated to the virial identity is globally coercive in $\mathcal{H}$.
- This will involve a detailed analysis of the phase-portrait for the Euler-Lagrange equations associated to the virial functional.

Virial Inequality

The key ingredient in the "rigidity argument" is the following virial inequality. In what follows $\chi_R(r) = \chi(\frac{r}{R})$ is a smooth cut-off function that is 1 on $[1, R]$ and zero for $r \geq 2R$. If $\vec{\psi} \in \mathcal{H}$ is a solution to (1), then for all $T \in \mathbb{R}$

$$\left\langle \chi_R \dot{\psi} \,|\, r\psi_r + (29/20)\psi \right\rangle \Big|_0^T \leq \int_0^T \mathcal{L}(\psi) + O\left(\mathcal{E}_R^\infty(\vec{\psi})\right)\, dt \quad (8)$$

Where $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{L}(\psi) := &- \int_1^\infty \left(\frac{1}{20}\dot{\psi}^2 + \frac{19}{20}\psi_r^2 \right) r^2\, dr \\ &+ \int_1^\infty \left(\sin^2(\psi) - \frac{29}{20}\psi \sin(2\psi) \right) dr\end{aligned}$$

When combined with the virial inequality, the following lemma is enough to prove that the only compact trajectory is $\psi \equiv 0$.

Lemma 1

Let $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ be defined as in the previous slide. Then for every $\vec{\psi} = (\psi(t), \dot{\psi}(t)) \in \mathcal{H}$ we have

$$\mathcal{L}(\vec{\psi}) \leq -\frac{1}{20} \int_1^\infty \left(\dot{\psi}^2 + \psi_r^2 \right) r^2 \, dr \leq -\frac{1}{180} \mathcal{E}(\vec{\psi})$$

- Lemma 1 means that the nonlinear virial functional $\mathcal{L}$ is globally coercive on the energy space.

Proof of Theorem 1

Indeed, applying Lemma 1 to the critical element $\vec{\psi}$ and plugging this into (8) gives

$$\left\langle \chi_R \dot{\psi} \,|\, r\psi_r + 29/20\psi \right\rangle \Big|_0^T \leq -\int_0^T \frac{\mathcal{E}(\vec{\psi})}{180} + O\left(\mathcal{E}_R^\infty(\vec{\psi})\right)\, dt \quad (9)$$

- By the pre-compactness K_+ we can choose R large enough so that $\mathcal{E}_R^\infty(\vec{\psi})$ is small uniformly in $t \geq 0$. Hence the right-hand-side of (9) is $\leq -cT\mathcal{E}(\vec{\psi})$.
- The left-hand-side of (9) is $O(R\mathcal{E}(\vec{\psi}))$. Hence, for every T we have

$$T\mathcal{E}(\vec{\psi}) \leq CR\mathcal{E}(\vec{\psi})$$

which is a contradiction since ψ is global. This proves Theorem 1. It remains to establish Lemma 1.

Proof of Lemma 1: Observe that

$$\mathcal{L}(\vec{\psi}) = -\frac{1}{20}\int_1^\infty \left(\dot{\psi}^2 + \psi_r^2\right) r^2\,dr + \Lambda(\psi)$$

where

$$\begin{aligned}\Lambda(\psi) := &-\frac{9}{10}\int_1^\infty \psi_r^2\, r^2\,dr + \int_1^\infty \left(\sin^2(\psi) - \frac{29}{20}\psi\sin(2\psi)\right)\,dr\\ = &-\frac{5}{9}E(\psi) + N(\psi)\end{aligned}$$

- It suffices to show that

$$\Lambda(\psi) \le 0 \quad \text{for every} \quad \psi \in \dot{H}_0^1(1,\infty) \tag{10}$$

- We prove (10) first on the subspace $\mathcal{A}_R := \dot{H}_0^1(1,R)$ for every R and then extend to all of $\dot{H}_0^1(1,\infty)$ by an approximation argument.

Euler-Lagrange equation

Again, we want to prove that

$$\Lambda(\psi) \leq 0 \quad \text{for every} \quad \psi \in \mathcal{A}_R \tag{11}$$

We claim that $\psi \equiv 0$ is the unique maximizer for $\Lambda|_{\mathcal{A}_R}$ for every R.

- After establishing the existence, we obtain the Euler-Lagrange equation for a maximizer.

$$\psi_{rr} + \frac{2}{r}\psi_r = \frac{1}{r^2} f(\psi) \tag{12}$$
$$\psi(1) = 0, \ \psi(R) = 0$$

where $f(x) := \frac{1}{4}\sin(2x) + \frac{29}{18} x \cos(2x)$.

- Setting $t = \log(r)$ and defining $x(t) := \psi(r)$ we obtain the following autonomous differential equation for x:

$$\ddot{x} + \dot{x} = f(x) \tag{13}$$
$$x(0) = 0, \ x(\log(R)) = 0$$

Euler-Lagrange equation

The proof now follows from the following lemma:

Lemma 2

Let $f(x) := \frac{1}{4}\sin(2x) + \frac{29}{18}x\cos(2x)$. Suppose that $x(t)$ is a solution to

$$\ddot{x} + \dot{x} = f(x) \tag{14}$$

and suppose that $x(0) = 0$ and that there exists a $T > 0$ such that $x(T) = 0$. Then $x \equiv 0$.

- We remark that the conclusion of Lemma 2 is extremely sensitive to the the exact form of f. Lemma 2 is false if f is replaced by $\frac{3}{2}f$.

Analysis of the Phase Portrait

Lemma 2 will be established via a detailed analysis of the phase portrait of (14). To begin, we set $y = \dot{x}$ and rewrite (14) as the following autonomous system:

$$\dot{v} := \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + f(x) \end{pmatrix} =: N(v) \qquad (15)$$

- Let x_j be the zeros of f. Note f, $-x_j = x_{-j}$. $v_j := (x_j, 0)$ are then fixed points of (15).
- Each v_j is a hyperbolic fixed point and one can show that
  1. v_j is a sink if j is odd
  2. v_j is a saddle if j is even

Phase Plane 1

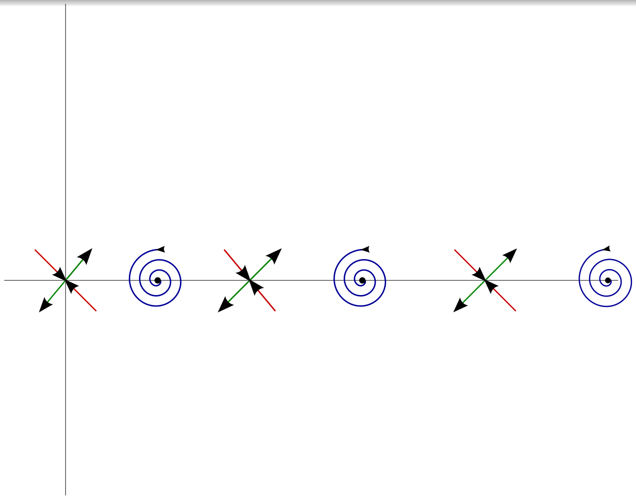

Figure: A depiction of the phase portrait associated to (15). The red and green flow lines correspond to the saddles at the fixed points v_j for j even and the blue flow lines represent the sinks at the v_j for j odd.

Phase Plane 2

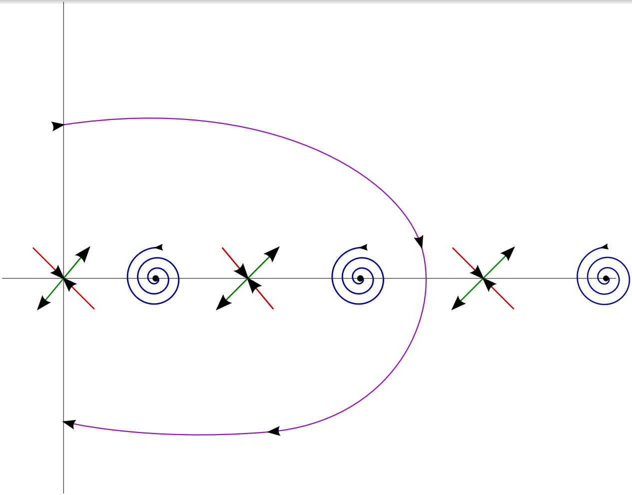

Figure: If Lemma 2 is false, then there would be a trajectory as depicted by the purple line in the above schematic with $v(0) = (0, v_0)$ and $v(T) = (0, v_1)$. Our goal is to rule out such a trajectory.

Phase Plane 3

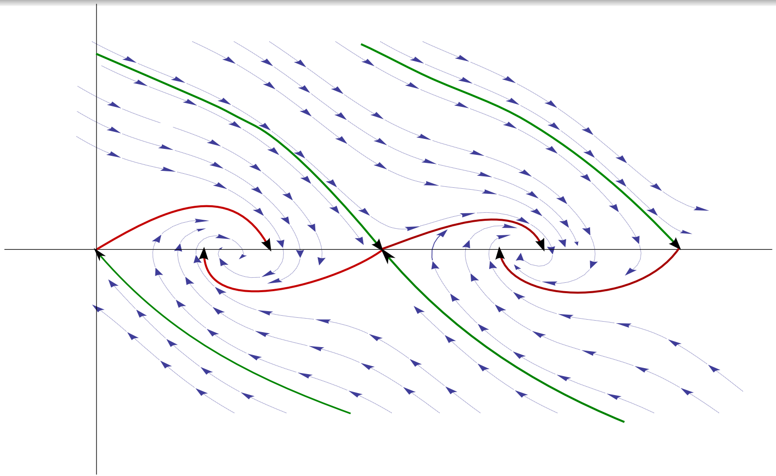

Figure: The figure above represents a slice of the phase portrait associated to (15). The red flow lines represent the unstable manifolds, W_j^u, associated to the v_j, and the green flow lines represent the stable manifolds, W_j^s, associated to the v_j.

Main tool

- Proving that the form of the red trajectories corresponding to the unstable manifolds as depicted in the previous slide is a delicate matter. For this we will need to construct suitable Lyapunov functionals. We will also need the following:

Key identity:

$$\frac{1}{2}(y^2(t_1) - y^2(t_0)) + \int_{t_0}^{t_1} y^2(s)\, ds = F(x(t_1)) - F(x(t_0)) \qquad (*)$$

where $F(x) := \frac{5}{18}\cos(2x) + \frac{29}{36}x\sin(2x)$ is a primitive for f. This is obtained by multiplying the equation (14) by $\dot{x}$ and integrating from t_0 to t_1.

- The form of the green trajectories corresponding to the stable manifolds is clear once we have established that the red trajectories have the desired form.

Lyapunov Functional

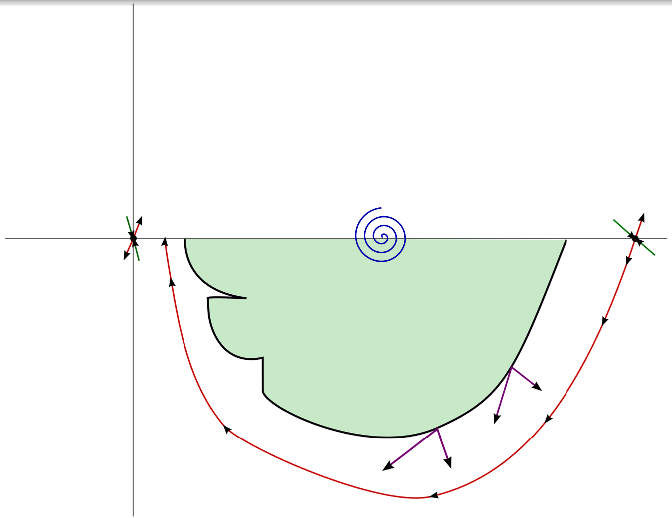

Figure: The red trajectory corresponds to the unstable manifold at v_2. The green region Σ is Lyapunov in the sense that $\partial\Sigma$ is repulsive with respect to the forward flow of our vector field (15)

Lyapunov Functional

- For the sake of finding a contradiction assume that the red trajectory does not fall into the sink at v_1. Then there exists a time T such that $v_2^-(T) = (0, y_2(T))$. Using the identity (*) with $t_0 = -\infty$, $t_1 = T$ we have

$$\frac{1}{2}y_2^2(T) + \int_{-\infty}^{T} y_2^2(s)\,ds = F(0) - F(x_2) < 2.18 \tag{16}$$

- Now we use the fact that since $\partial\Sigma$ is Lyapunov the trajectory v_2^- cannot enter Σ and hence the integral on the left-hand-side of (16) is greater than the area of Σ, i.e.,

$$2.21 < \mathrm{Area}(\Sigma) < \int_{-\infty}^{T} y_2^2(s)\,ds \tag{17}$$

- $2.18 < 2.21$ but only by $.03$!

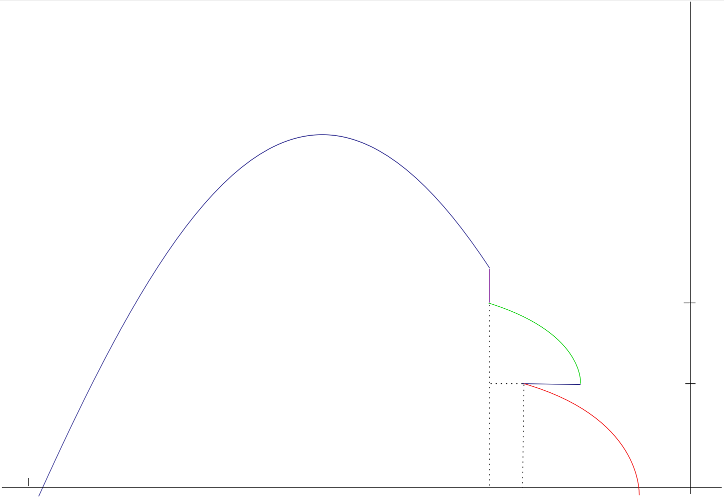

Figure: The region $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ pictured above has the property that $\partial\Sigma$ is repulsive with respect to the unstable manifold W^u_{-2}.

Construction of $\partial\Sigma$

To construct Σ, we define three polynomials, p_1, p_2, p_3. As an example the first polynomial p_1 as a function of x is defined by:

$$p_1(x) := -\frac{3}{1000} + \frac{110}{47}\left(x+\frac{43}{18}\right) - \frac{89}{222}\left(x+\frac{43}{18}\right)^2 - \frac{23}{42}\left(x+\frac{43}{18}\right)^3 + \frac{7}{85}\left(x+\frac{43}{18}\right)^4 + \frac{8}{303}\left(x+\frac{43}{18}\right)^5 - \frac{1}{446}\left(x+\frac{43}{18}\right)^6 - \frac{1}{760}\left(x+\frac{43}{18}\right)^7 + \frac{1}{4035}\left(x+\frac{43}{18}\right)^8 - \frac{1}{13999}\left(x+\frac{43}{18}\right)^9$$

We set $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Σ_1 is defined by

$$\Sigma_1 := \left\{(x,y) \in \Omega_{-1} \,\middle|\, -\frac{43}{18} + \frac{3}{1000} < x < -\frac{3}{5},\ 0 < y < p_1\left(-\frac{3}{5}\right)\right\}$$

Numerical simulation of flow in first strip

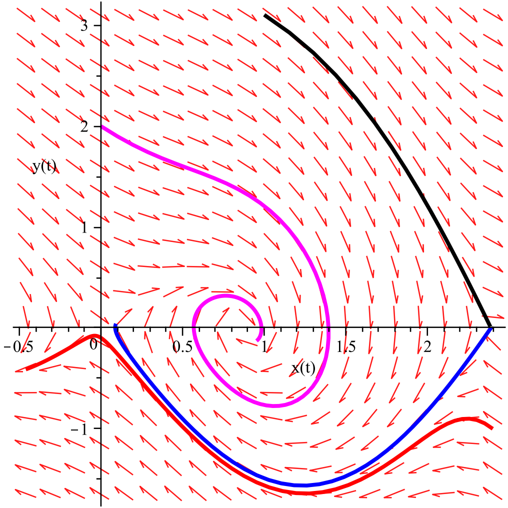

Figure: A schematic depiction of the flow in the first strip using Maple.

Numerical simulation of flow in second strip

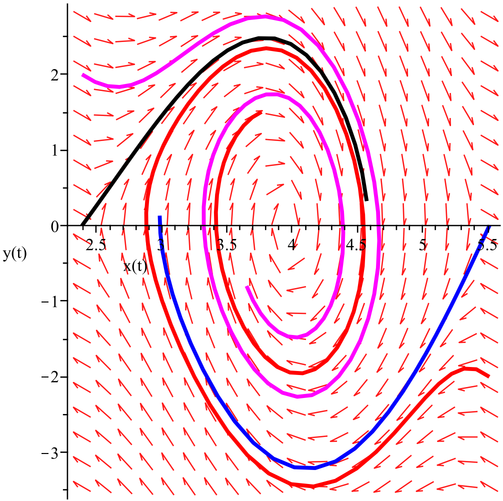

Figure: A schematic depiction of the flow in the second strip using Maple.

Renormalization

To deal with the trajectories emanating from the v_j for even $j > 4$ we shift and rescale our equation (15) via the following renormalization. For each $j \in \mathbb{N}$, $\varepsilon \in \mathbb{R}$ we define ζ and η via

$$\begin{aligned} x(t) &=: \frac{2j-1}{4}\pi + \zeta(\varepsilon^{-1}t) \\ y(t) &=: \varepsilon^{-1}\,\eta(\varepsilon^{-1}t) \end{aligned} \tag{18}$$

Define $z_j := \frac{2j-1}{4}\pi$. Then (15) implies the following system of equations for ζ, η

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \eta \\ -\varepsilon\eta + \varepsilon^2 f(z_j + \zeta) \end{pmatrix} \tag{19}$$

where $\dot{} = \frac{d}{ds}$ where $s = \varepsilon^{-1} t$.

Renormalization 2

Set $g(\zeta) := \frac{1}{4}\cos(2\zeta) - \frac{29}{18}\zeta\sin(2\zeta)$. And for even $j > 4$ set

$$\varepsilon := \sqrt{\frac{72}{29\pi(2j-1)}}$$

Observe $\varepsilon < \frac{7}{20}$ for $j \geq 6$. Then(19) becomes

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \eta \\ \sin(2\zeta) - \varepsilon\eta - \varepsilon^2 g(\zeta) \end{pmatrix} \tag{20}$$

- Note that (20) is the equation governing the motion of a damped pendulum with a small perturbative term $\varepsilon^2 g(\zeta)$, and in the limit as $\varepsilon \to 0$, (20) is exactly the the equation of a simple pendulum.
- After this renormalization, the proof follows the same general outline–phase plane analysis, construction of a Lyapunov functional–as for the first two strips.

Renormalized Phase Plane

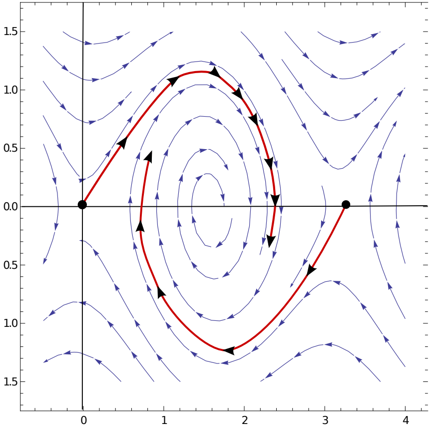

Figure: A schematic depiction of the flow for the renormalized equation.

Reminder of Main Results

Theorem 1 (Lawrie, S., 2011)

$n = 0$, $Q_0 = 0$. For any smooth energy data $(\psi_0, \psi_1) \in \mathcal{H}$, there exists a global smooth evolution $\vec{\psi}$ to (1). Furthermore, $\vec{\psi}$ scatters to 0 in the sense that the energy of $\vec{\psi}$ on any arbitrary, but compact region vanishes as $t \to \infty$.

Theorem 2 (Lawrie, S., 2011)

$n \geq 1$, $Q_n = Q$. There exists $\varepsilon > 0$ such that for all smooth data $(\psi_0, \psi_1) \in \mathcal{H}_n$ such that

$$\|(\psi_0, \psi_1) - (Q, 0)\|_{\mathcal{H}} < \varepsilon$$

the unique solution ψ to (1) with data (ψ_0, ψ_1) exists globally in time and scatters to Q as $t \to \infty$.

Remarks regarding Theorem 2

- Theorem 2 is proved by establishing Strichartz estimates for the wave equation exterior to a ball perturbed by a radial potential V which arises from the linearization of the problem about the harmonic map Q_n. To be precise we prove Strichartz estimates for

$$(\partial_{tt} - \Delta_5 + V)u = F$$
$$u(t,1) = 0, \quad (u(0), u_t(0)) = (u_0, u_1) \in \dot{H}^1_0 \times L^2(\mathbb{R}^5_*, \mathrm{radial})$$
$$V(r) = \frac{2}{r^2}(\cos(2Q(r)) - 1)$$

  One can show that $Q_n(r) = n\pi - O(r^{-2})$ as $r \to \infty$ so V decays like r^{-6} as $r \to \infty$.
- The idea is to extend the exterior Strichartz estimates of Metcalfe, Smith, and Sogge to this setting via local energy estimates. It is crucial that the operator $-\Delta + V$ has no negative spectrum, and no eigenvalue or resonance at 0.

- Conjecture of Bizon, Chmaj, Maliborksi: Numerical simulations suggest that Theorem 2 holds with $\varepsilon = \infty$.
- This currently appears out of reach. The main difficulty with the implementation of the Kenig-Merle method lies with the coercivity of the virial functional centered at the harmonic maps Q_n.
- Euler-Lagrange equations involve Q_n, hence cannot be transformed into an autonomous system
- Q_n is not explicit.

The End

Thank you!
p.s. the slides from this talk can be found on my webpage:
math.uchicago.edu/∼alawrie